

# ASYMPTOTIC CONE OF SEMISIMPLE ORBITS FOR SYMMETRIC PAIRS

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**ABSTRACT.** Let  $G$  be a reductive algebraic group over  $\mathbb{C}$  and denote its Lie algebra by  $\mathfrak{g}$ . Let  $\mathbb{O}_h$  be a closed  $G$ -orbit through a semisimple element  $h \in \mathfrak{g}$ . By a result of Borho and Kraft [BK79], it is known that the asymptotic cone of the orbit  $\mathbb{O}_h$  is the closure of a Richardson nilpotent orbit corresponding to a parabolic subgroup whose Levi component is the centralizer  $Z_G(h)$  in  $G$ . In this paper, we prove an analogue on a semisimple orbit for a symmetric pair.

More precisely, let  $\theta$  be an involution of  $G$ , and  $K = G^\theta$  a fixed point subgroup of  $\theta$ . Then we have a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$  of the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$  which is the eigenspace decomposition of  $\theta$  on  $\mathfrak{g}$ . Let  $\{x, h, y\}$  be a normal  $\mathfrak{sl}_2$  triple, where  $x, y \in \mathfrak{s}$  is nilpotent, and  $h \in \mathfrak{k}$  semisimple. In addition, we assume  $\bar{x} = y$ , where  $\bar{x}$  denotes the complex conjugation which commutes with  $\theta$ . Then  $a = \sqrt{-1}(x - y)$  is a semisimple element in  $\mathfrak{s}$ , and we can consider a semisimple orbit  $\text{Ad}(K)a$  in  $\mathfrak{s}$ , which is closed. Our main result asserts that the asymptotic cone of  $\text{Ad}(K)a$  in  $\mathfrak{s}$  coincides with  $\overline{\text{Ad}(G)x \cap \mathfrak{s}}$ , if  $x$  is even nilpotent.

## INTRODUCTION

Let  $G$  be a connected reductive algebraic group over  $\mathbb{C}$  and denote its Lie algebra by  $\mathfrak{g}$ . Let  $h \in \mathfrak{g}$  be a semisimple element and denote by  $\mathbb{O}_h$  the adjoint  $G$ -orbit through  $h$ . It is a closed affine subvariety in  $\mathfrak{g}$ . With this semisimple orbit, we can associate two objects.

One object is a nilpotent orbit called a Richardson orbit. To be more precise, let us consider the centralizer  $L := Z_G(h)$  of  $h$ . Then, there is a parabolic subgroup  $P$  whose Levi component is  $L$ . Let us denote a Levi decomposition of the Lie algebra  $\mathfrak{p}$  by  $\mathfrak{l} + \mathfrak{u}$ , where  $\mathfrak{u}$  denotes the nilpotent radical of  $\mathfrak{p}$ . Then  $\text{Ad}(G)\mathfrak{u}$  is the closure of a single nilpotent orbit  $\mathcal{O}$ , which is called the Richardson orbit associated with  $P$ . The Richardson orbit  $\mathcal{O}$  in fact does not depend on the choice of the parabolic  $P$ , and it is determined by  $h$ .

The other object, which we consider, is the asymptotic cone  $\mathfrak{C}(\mathbb{O}_h)$  of  $\mathbb{O}_h$ , which indicates the asymptotic direction in which the variety  $\mathbb{O}_h$  spreads out. See §1 for precise definition.

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In [BK79], Borho and Kraft studied Dixmier sheets, and in the course of their study they proved the following theorem.

**Theorem 0.1** (Borho-Kraft). *For a semisimple orbit  $\mathbb{O}_h$ , the asymptotic cone  $\mathfrak{C}(\mathbb{O}_h)$  coincides with the closure of the Richardson nilpotent orbit  $\overline{\mathcal{O}}$  above.*

This can be interpreted as a generalization of Kostant's theorem, which asserts that the nilpotent variety  $\mathcal{N}(\mathfrak{g})$  is a deformation of the regular semisimple orbits ([Kos63]). Note that  $\mathcal{N}(\mathfrak{g})$  is the closure of a principal nilpotent orbit, which is a Richardson orbit associated with a Borel subgroup. In this case, the “deformation” amounts to taking an asymptotic cone of regular semisimple orbits.

In this paper, we prove an analogous theorem for a semisimple orbit for a symmetric pair.

Let us explain it more precisely. Let  $\theta$  be an involution of  $G$ , and  $K = G^\theta$  a fixed point subgroup of  $\theta$ . Then we have a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$  of the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$  which is the eigenspace decomposition of  $\theta$  on  $\mathfrak{g}$ . We pick a nilpotent element  $x$  in  $\mathfrak{s}$ , and consider a normal  $\mathfrak{sl}_2$  triple  $\{x, h, y\}$ , where  $x, y \in \mathfrak{s}$  is nilpotent, and  $h \in \mathfrak{k}$  semisimple. In addition, we can assume  $\bar{x} = y$  without loss of generality, where  $\bar{x}$  denotes the complex conjugation which commutes with  $\theta$ . Then  $a = \sqrt{-1}(x - y)$  is a semisimple element in  $\mathfrak{s}_\mathbb{R}$ , and we can consider a semisimple orbit  $\mathbb{O}_a^K = \text{Ad}(K)a$  in  $\mathfrak{s}$ , which is closed.

Our main result asserts that, if  $x$  is even nilpotent, the asymptotic cone of  $\mathbb{O}_a^K$  in  $\mathfrak{s}$  coincides with  $\overline{\mathbb{O}_x^G \cap \mathfrak{s}}$ , where  $\mathbb{O}_x^G = \text{Ad}(G)x$  is a nilpotent  $G$ -orbit through  $x$ . In fact, the intersection  $\mathbb{O}_x^G \cap \mathfrak{s}$  breaks up into several nilpotent  $K$ -orbits,

$$\mathbb{O}_x^G \cap \mathfrak{s} = \bigcup_{i=0}^{\ell} \mathbb{O}_{x_i}^K,$$

each of which is a Lagrangian subvariety of  $\mathbb{O}_x^G$ . So we can state our main theorem as

**Theorem 0.2.** *Suppose  $x \in \mathfrak{s}$  is an even nilpotent element, and construct a semisimple element  $a \in \mathfrak{s}_\mathbb{R}$  as explained above. Then the asymptotic cone of the semisimple orbit  $\mathbb{O}_a^K$  in  $\mathfrak{s}$  is given by*

$$\mathfrak{C}(\mathbb{O}_a^K) = \overline{\mathbb{O}_x^G \cap \mathfrak{s}} = \bigcup_{i=0}^{\ell} \overline{\mathbb{O}_{x_i}^K}.$$

Note that the asymptotic cone is no longer irreducible in the case of symmetric pair. This reflects the reducibility of the nilpotent variety for symmetric pairs as pointed out by [KR71]. Our theorem can be seen as a generalization of Kostant-Rallis's theorem.

From the semisimple element  $a \in \mathfrak{s}_\mathbb{R}$ , we can construct a real parabolic subgroup  $P_\mathbb{R}$  in a standard way (see §4). The asymptotic cone above is the associated variety of a degenerate principal series representation  $\text{Ind}_{P_\mathbb{R}}^{G_\mathbb{R}} \chi$  induced from a character  $\chi$  of  $P_\mathbb{R}$ . It seems that the irreducible components  $\mathbb{O}_{x_i}^K$  of  $\mathfrak{C}(\mathbb{O}_a^K)$  play an important role in the theory of degenerate principal series representations. We discuss what we can expect for this, using an example in the case of  $G_\mathbb{R} = U(n, n)$  in §5.

## 1. ASYMPTOTIC CONE

Let  $V = \mathbb{C}^N$  be a vector space. For a subvariety  $X \subset V$ , we define the asymptotic cone of  $X$ , denoted by  $\mathfrak{C}^{\mathbb{P}}(X) \subset \mathbb{P}(V)$ , as follows. We extend  $V$  by the one-dimensional vector space, and denote it by  $\tilde{V} = V \oplus \mathbb{C}$ . We consider the projective space  $\mathbb{P}(\tilde{V})$ . Then there is a natural open embedding  $\iota : V \hookrightarrow \mathbb{P}(\tilde{V})$  defined by  $\iota(v) = [v \oplus 1]$ , where  $[w]$  denotes the image of  $w \in \tilde{V} \setminus \{0\}$  in  $\mathbb{P}(\tilde{V})$  under the natural projection. On the other hand, there is a closed embedding  $\kappa : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\tilde{V})$  which send  $[u] \in \mathbb{P}(V)$  to  $\kappa([u]) := [u \oplus 0] \in \mathbb{P}(\tilde{V})$ . Thus we have a disjoint decomposition  $\mathbb{P}(\tilde{V}) = \iota(V) \sqcup \kappa(\mathbb{P}(V))$ . In the following, we identify  $\mathbb{P}(V)$  with  $\kappa(\mathbb{P}(V))$  and consider it as a closed subvariety of  $\mathbb{P}(\tilde{V})$ .

**Definition 1.1.** Let  $X$  be a subvariety of  $V$  of positive dimension. We define the asymptotic cone of  $X$  by  $\mathfrak{C}^{\mathbb{P}}(X) := \overline{\iota(X)} \cap \mathbb{P}(V)$ , where  $\mathbb{P}(V)$  is identified with  $\kappa(\mathbb{P}(V)) \subset \mathbb{P}(\tilde{V})$ . Then  $\mathfrak{C}^{\mathbb{P}}(X) \subset \mathbb{P}(V)$  is a projective variety of the same dimension as  $X$ . The affine cone in  $V$  associated to  $\mathfrak{C}^{\mathbb{P}}(X)$  is denoted by  $\mathfrak{C}(X)$ , and we call it the *affine asymptotic cone*, while  $\mathfrak{C}^{\mathbb{P}}(X)$  is called the *projective asymptotic cone*.

If  $X$  is 0-dimensional, i.e., if it consists of a finite set of points, we put  $\mathfrak{C}^{\mathbb{P}}(X) = \emptyset$  and  $\mathfrak{C}(X) = \{0\}$ .

The asymptotic cone was introduced by W. Borho and H. Kraft ([BK79]) to study Dixmier sheets of the adjoint representation of a reductive algebraic group. We refer the readers to [BK79] for the details of their properties. Here in this section we only recall some properties of asymptotic cones without proof.

Let  $I$  be an ideal of the polynomial ring  $\mathbb{C}[V]$ . For  $f \in I$ , let  $\text{gr } f$  be the homogeneous part of the maximal degree. We define  $\text{gr } I = (\text{gr } f \mid f \in I)$ , the homogeneous ideal generated by  $\text{gr } f$  ( $f \in I$ ).

Let  $\mathbb{I}(X)$  be the annihilator ideal of  $X$ . Then the annihilator ideal of the asymptotic cone is given by  $\mathbb{I}(\mathfrak{C}(X)) = \sqrt{\text{gr } \mathbb{I}(X)}$ . Thus the regular function ring  $\mathbb{C}[\mathfrak{C}(X)]$  is isomorphic to  $\mathbb{C}[V]/\sqrt{\text{gr } \mathbb{I}(X)}$ , which is equal to the homogeneous function ring of  $\mathfrak{C}^{\mathbb{P}}(X)$ .

Let  $G$  be a connected reductive algebraic group over  $\mathbb{C}$  which acts linearly on  $V$  and assume that  $X$  is stable under  $G$ . Then the ring of regular functions  $\mathbb{C}[X]$  has a natural  $G$ -module structure. The asymptotic cone  $\mathfrak{C}^{\mathbb{P}}(X)$  as well as  $\mathfrak{C}(X)$  is also a  $G$ -variety, and we have a  $G$ -action on the regular function ring  $\mathbb{C}[\mathfrak{C}(X)]$  in particular.

**Lemma 1.2.** *Let  $X$  be a closed affine variety in  $V$  which is stable under the action of  $G$ , and  $I = \mathbb{I}(X)$  an annihilator ideal of  $X$ . Then  $\mathbb{C}[X] \simeq \mathbb{C}[V]/I$  is isomorphic to  $\mathbb{C}[V]/\text{gr } I$  as a  $G$ -module. Since  $\mathbb{C}[\mathfrak{C}(X)] \simeq \mathbb{C}[V]/\sqrt{I}$ , we have a surjective  $G$ -module morphism  $\mathbb{C}[X] \rightarrow \mathbb{C}[\mathfrak{C}(X)]$ .*

Let  $\mathfrak{N}(V) := \{v \in V \mid f(v) = 0 \ (f \in \mathbb{C}[V]_+^G)\}$  be the null fiber. It is the zero locus of homogeneous  $G$ -invariants of positive degree.

**Proposition 1.3.** *Let  $\mathbb{O}$  be a  $G$ -orbit in  $V$ . Then the affine asymptotic cone  $\mathfrak{C}(\mathbb{O})$  is a  $G$ -stable subvariety of  $\mathfrak{N}(V)$ , which is equidimensional and  $\dim \mathfrak{C}(\mathbb{O}) = \dim \mathbb{O}$ .*

Let  $\mathfrak{g}$  be a Lie algebra on which  $G$  acts by the adjoint action. Then the null fiber  $\mathfrak{N}(\mathfrak{g})$  is called the nilpotent variety, which consists of all the nilpotent elements in  $\mathfrak{g}$ . It is well known that  $\mathfrak{N}(\mathfrak{g})$  contains only a finite number of  $G$ -orbits.

**Corollary 1.4.** *For  $x \in \mathfrak{g}$ , let  $\mathbb{O}_x = \text{Ad}(G)x$  be the adjoint orbit through  $x$ . Then the affine asymptotic cone  $\mathfrak{C}(\mathbb{O}_x)$  is a finite union of the closure of nilpotent orbits, whose dimension is equal to  $\dim \mathbb{O}_x$ .*

In the following, we will denote the adjoint action simply by  $gx = \text{Ad}(g)x$  for  $g \in G$ ,  $x \in \mathfrak{g}$ .

## 2. RICHARDSON ORBIT

Let  $h \in \mathfrak{g}$  be a semisimple element, and put  $L := Z_G(h)$  the centralizer of  $h$  in  $G$ . There is a parabolic subgroup  $P$  with a Levi decomposition  $P = LU$ , where  $U$  is the unipotent radical. Then  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$  is a Levi decomposition of the corresponding Lie algebra.

**Definition 2.1.** Let  $\mathfrak{u}$  be the nilpotent radical of a parabolic subalgebra  $\mathfrak{p}$ . Then adjoint translate  $G\mathfrak{u} = \{\text{Ad}(g)u \mid g \in G, u \in \mathfrak{u}\}$  of  $\mathfrak{u}$  is the closure of a single nilpotent orbit  $\overline{\mathbb{O}_x}$  ( $x$  : nilpotent element). We call  $\mathbb{O}_x$  the *Richardson orbit* for the parabolic  $P$ , and  $x$  a *Richardson element*. We often assume  $x$  to be taken from  $\mathfrak{u}$ .

Let us consider a partial flag variety  $\mathfrak{B}_P := G/P$  of all parabolics conjugate to  $\mathfrak{p}$ , and denote by  $T^*\mathfrak{B}_P$  the cotangent bundle over  $\mathfrak{B}_P$ . Then there is a  $G$ -equivariant map  $\mu$  called the *moment map* defined as follows.

$$\mu : T^*\mathfrak{B}_P \simeq G \times_P \mathfrak{u} \ni (g, z) \rightarrow \text{Ad}(g)z \in \mathfrak{g}$$

The following proposition is well known. See [Jan04] and references therein.

**Proposition 2.2.** *Assume that  $x$  is a Richardson element for  $P$  and that  $Z_G(x) = Z_P(x)$  holds.*

- (1) *The moment map  $\mu : T^*\mathfrak{B}_P \rightarrow \overline{\mathbb{O}_x}$  is a resolution of singularities of  $\overline{\mathbb{O}_x}$ .*
- (2) *The fiber of  $\mathbb{O}_x$  is  $\mu^{-1}(\mathbb{O}_x) = G[e, x]$  and  $\mu : G[e, x] \xrightarrow{\sim} \mathbb{O}_x$  is an isomorphism.*
- (3) *The moment map  $\mu$  induces a  $G$ -equivariant isomorphism  $\mathbb{C}[G \times_P \mathfrak{u}] = \mathbb{C}[G \times \mathfrak{u}]^P \simeq \mathbb{C}[\mathbb{O}_x]$ . In addition, if  $\overline{\mathbb{O}_x}$  is normal, then  $\mathbb{C}[\overline{\mathbb{O}_x}] = \mathbb{C}[\mathbb{O}_x]$  holds.*

If a reductive group  $K$  acts on a variety  $\mathfrak{X}$ , we get a decomposition of the regular function ring as a  $K$ -module,

$$\mathbb{C}[\mathfrak{X}] \simeq \bigoplus_{\tau \in \text{Irr}(K)} m_{\tau}(\mathfrak{X}) \tau \quad (\text{as a } K\text{-module}), \quad (2.1)$$

where  $m_{\tau}(\mathfrak{X})$  denotes the multiplicity.

**Theorem 2.3** (Borho-Kraft). *Let  $h \in \mathfrak{g}$  be a semisimple element and define the parabolic subgroup  $P$  and the Richardson orbit  $\mathbb{O}_x$  as above. Then the asymptotic cone of the*

semisimple orbit  $\mathbb{O}_h$  is equal to the Richardson orbit :  $\mathfrak{C}(\mathbb{O}_h) = \overline{\mathbb{O}_x}$ . In addition, if  $Z_G(x)$  is connected and  $\overline{\mathbb{O}_x}$  is normal, we have

$$\mathbb{C}[\mathbb{O}_h] \simeq \text{Ind}_L^G \mathbf{1}_L \simeq \mathbb{C}[\mathbb{O}_x] = \mathbb{C}[\overline{\mathbb{O}_x}] = \mathbb{C}[\mathfrak{C}(\mathbb{O}_h)] \quad (\text{as } G\text{-modules})$$

i.e.,  $m_\tau(\mathbb{O}_h) = m_\tau(\mathbb{O}_x) = m_\tau(\mathfrak{C}(\mathbb{O}_h)) = \dim \tau^L \ (\forall \tau \in \text{Irr}(G))$ .

Up to this point, we started with a semisimple element, but now we investigate in other ways. So take a nilpotent element  $x \in \mathfrak{g}$ , and choose an  $\mathfrak{sl}_2$  triple  $\{x, h, y\}$ , where  $h$  is semisimple;  $x, y$  are nilpotent; and they satisfy the commutation relations

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Thus  $\mathfrak{g}$  is a representation space of  $\mathfrak{sl}_2 = \text{span}_{\mathbb{C}}\{x, h, y\}$ . Therefore the eigenvalues of  $\text{ad } h$  are integers and we get a  $\mathbb{Z}$ -grading of  $\mathfrak{g}$  induced by the action of  $\text{ad } h$ .

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k \quad \mathfrak{g}_k := \{X \in \mathfrak{g} \mid \text{ad}(h)X = kX\} \quad (2.2)$$

**Definition 2.4.** If  $\mathfrak{g}_1 = \{0\}$ ,  $x$  is called an *even* nilpotent element. Note that  $\mathfrak{g}_1 = \{0\}$  if and only if  $\mathfrak{g}_k = \{0\}$  ( $\forall k : \text{odd}$ ).

We put  $\mathfrak{p} = \bigoplus_{k \geq 0} \mathfrak{g}_k = \mathfrak{l} \oplus \mathfrak{u}$ , where  $\mathfrak{l} = \mathfrak{g}_0$  and  $\mathfrak{u} = \bigoplus_{k > 0} \mathfrak{g}_k$ . Then  $\mathfrak{p}$  is a parabolic subalgebra and, if  $x$  is even nilpotent, then  $\mathbb{O}_x$  is a Richardson orbit for  $P = N_G(\mathfrak{p})$ . Even nilpotent elements have good properties (see [Jan04] for example).

**Proposition 2.5.** Assume  $x$  is even nilpotent, then  $Z_G(x) = Z_P(x)$  holds. Hence the moment map  $\mu : T^*\mathfrak{B}_P \rightarrow \overline{\mathbb{O}_x}$  is a resolution of singularities, and we have an isomorphism of regular function rings  $\mathbb{C}[T^*\mathfrak{B}_P] \simeq \mathbb{C}[\mathbb{O}_x]$ .

Moreover, if  $\overline{\mathbb{O}_x}$  is normal, then  $\mathbb{C}[\overline{\mathbb{O}_x}] \simeq \mathbb{C}[\mathbb{O}_x] \simeq \mathbb{C}[T^*\mathfrak{B}_P]$ .

**Corollary 2.6.** Let  $\{x, h, y\}$  be an  $\mathfrak{sl}_2$  triple with  $x$  even nilpotent and assume that  $\overline{\mathbb{O}_x}$  is normal. Then the asymptotic cone of a semisimple element  $h$  is equal to the closure of the nilpotent orbit through  $x$ .

$$\mathfrak{C}(\mathbb{O}_h) = \overline{\mathbb{O}_x}$$

Moreover, there is an isomorphism  $\mathbb{C}[\mathfrak{C}(\mathbb{O}_h)] \simeq \mathbb{C}[T^*\mathfrak{B}_P]$ .

### 3. RICHARDSON ORBIT FOR SYMMETRIC PAIR

Let  $G_{\mathbb{R}}$  be a reductive Lie group, which is a real form of a connected complex algebraic group  $G$ . We fix a Cartan involution  $\theta$ . Then the fixed point subgroup of  $\theta$  is a maximal compact subgroup  $K_{\mathbb{R}} = G_{\mathbb{R}}^{\theta}$ . We extend  $\theta$  to  $G$  holomorphically, and put  $K = G^{\theta}$ , which is a complexification of  $K_{\mathbb{R}}$ . We mainly consider a symmetric pair  $(G, K)$  in the following.

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$  a (complexified) Cartan decomposition, where  $\mathfrak{k}$  is the Lie algebra of  $K$  and  $\mathfrak{s}$  is the  $(-1)$ -eigenspace of the differential of  $\theta$ .

Take a  $\theta$ -stable parabolic subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$ . We denote by  $P$  the corresponding parabolic subgroup of  $G$ , and put  $\mathfrak{B}_P = G/P$ , the partial flag variety. Then  $\mathfrak{B}_P$  can be considered as the totality of the parabolic subalgebras of  $\mathfrak{g}$  which is conjugate to  $\mathfrak{p}$  by the adjoint

action of  $G$ . The  $K$ -orbit of the  $\theta$ -stable parabolic  $\mathfrak{p}$  is a closed orbit in  $\mathfrak{B}_P$ . Conversely, if there is a  $\theta$ -stable parabolic, then any closed  $K$ -orbit in  $\mathfrak{B}_P$  arises as a  $K$ -conjugacy class of  $\theta$ -stable parabolic subalgebras.

Let  $\mathcal{O}$  denote a closed  $K$ -orbit in  $\mathfrak{B}_P$  generated by  $\mathfrak{p}$ . Then the conormal bundle  $T_{\mathcal{O}}^*\mathfrak{B}_P$  over  $\mathcal{O}$  can be described as follows.

Since  $\mathfrak{p}$  is  $\theta$ -stable,  $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{k}$  is a parabolic subalgebra in  $\mathfrak{k}$ . Let  $Q$  be the corresponding parabolic subgroup of  $K$ . If  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$  is a  $\theta$ -stable Levi decomposition,  $\mathfrak{q} = \mathfrak{l}(\mathfrak{k}) \oplus \mathfrak{u}(\mathfrak{k})$  with  $\mathfrak{l}(\mathfrak{k}) = \mathfrak{l} \cap \mathfrak{k}$  and  $\mathfrak{u}(\mathfrak{k}) = \mathfrak{u} \cap \mathfrak{k}$  gives a Levi decomposition of  $\mathfrak{q}$ . Also we put  $\mathfrak{u}(\mathfrak{s}) = \mathfrak{u} \cap \mathfrak{s}$ . Then  $\mathfrak{u}(\mathfrak{s})$  is  $Q$ -stable, and we have

$$T_{\mathcal{O}}^*\mathfrak{B}_P \simeq K \times_Q \mathfrak{u}(\mathfrak{s}) = (K \times \mathfrak{u}(\mathfrak{s}))/Q$$

where the action of  $Q$  on  $K \times \mathfrak{u}(\mathfrak{s})$  is given by  $q(k, x) = (kq^{-1}, \text{Ad}(q)x)$  for  $q \in Q, k \in K, x \in \mathfrak{u}(\mathfrak{s})$ . We denote the class of  $(k, x) \in K \times \mathfrak{u}(\mathfrak{s})$  in  $K \times_Q \mathfrak{u}(\mathfrak{s})$  by  $[k, x]$ . Then a map

$$\mu : T_{\mathcal{O}}^*\mathfrak{B}_P \simeq K \times_Q \mathfrak{u}(\mathfrak{s}) \rightarrow \mathfrak{s}, \quad \mu([k, x]) = \text{Ad}(k)x$$

is well-defined, and called the *moment map*. For any  $K$ -orbit  $\mathcal{O}$  in  $\mathfrak{B}_P$ , the moment map image of the conormal bundle  $T_{\mathcal{O}}^*\mathfrak{B}_P$  is the closure of a single nilpotent  $K$ -orbit  $\mathbb{O}^K$  in  $\mathfrak{s}$ . The following definition is due to P. Trape [Tra05] (see also [Tra07]).

**Definition 3.1.** Let  $\mathfrak{p}$  be a  $\theta$ -stable parabolic subalgebra and  $\mathcal{O}$  a closed  $K$ -orbit in  $\mathfrak{B}_P$  through  $\mathfrak{p}$ . If a nilpotent  $K$ -orbit  $\mathbb{O}^K \subset \mathfrak{s}$  is dense in the moment map image of  $T_{\mathcal{O}}^*\mathfrak{B}_P$ , it is called a *Richardson orbit for the symmetric pair  $G/K$*  associated to  $\mathfrak{p}$ .

The following is a representation theoretic characterization of Richardson orbits.

**Theorem 3.2.** A nilpotent  $K$ -orbit  $\mathbb{O}^K \subset \mathfrak{s}$  is a Richardson orbit for the symmetric pair if and only if its closure is the associated variety of a derived functor module  $A_{\mathfrak{p}}$  with the trivial infinitesimal character for a certain  $\theta$ -stable parabolic subalgebra  $\mathfrak{p}$ .

#### 4. ASYMPTOTIC CONE FOR SYMMETRIC PAIR

Let  $x \in \mathfrak{s}$  be a nilpotent element. Then we can choose  $y \in \mathfrak{s}$  and  $h \in \mathfrak{k}$  such that  $\{x, h, y\}$  forms a normal  $\mathfrak{sl}_2$  triple, where  $x, y$  are nilpotent, and  $h$  semisimple (see [CM93, §9.4] for example). In addition, after suitable conjugation by  $K$ , we can assume  $\bar{x} = y$ , where  $\bar{x}$  denotes the complex conjugation with respect to  $\mathfrak{g}_{\mathbb{R}}$ . We call a normal  $\mathfrak{sl}_2$  triple with this property a *KS triple*. Then

$$a = \sqrt{-1}(x - y) \in \mathfrak{s}_{\mathbb{R}}$$

is a semisimple element in  $\mathfrak{s}_{\mathbb{R}}$ . Also we put

$$e = \frac{1}{2}(x + y + \sqrt{-1}h), \quad f = \frac{1}{2}(x + y - \sqrt{-1}h) = -\theta(e).$$

Then  $e$  and  $f$  are nilpotent elements belonging to the real form  $\mathfrak{g}_{\mathbb{R}}$ , and  $\{e, a, f\}$  is a standard  $\mathfrak{sl}_2$  triple in  $\mathfrak{g}_{\mathbb{R}}$ . We call it a *Cayley triple*. Every standard  $\mathfrak{sl}_2$  triple is  $G_{\mathbb{R}}$ -conjugate to a Cayley triple.



The following theorem is well known.

**Theorem 4.1** (Sekiguchi [Sek87], Vergne [Ver95]). *Nilpotent orbits  $\mathbb{O}_x^K = \text{Ad}(K)x$  and  $\mathbb{O}_e^{G_{\mathbb{R}}} = \text{Ad}(G_{\mathbb{R}})e$  are  $K_{\mathbb{R}}$ -equivariantly diffeomorphic, and moreover they generate the same nilpotent  $G$ -orbit:  $\text{Ad}(G)x = \text{Ad}(G)e$ . This correspondence gives a bijection between the set of non-zero nilpotent  $K$ -orbits in  $\mathfrak{s}$  and that of non-zero nilpotent  $G_{\mathbb{R}}$ -orbits in  $\mathfrak{g}_{\mathbb{R}}$ .*

See [CM93, Theorem 9.5.1 & Remark 9.5.2] and [BS98] for further properties.

Let us denote  $\mathbb{O}_x^G = \text{Ad}(G)x$ . Then the intersection  $\mathbb{O}_x^G \cap \mathfrak{s}$  breaks up into several nilpotent  $K$ -orbits  $\bigcup_{i=0}^{\ell} \mathbb{O}_{x_i}^K$  where  $x = x_0$ . It is well known that each  $\mathbb{O}_{x_i}^K$  is a Lagrangian subvariety for the canonical symplectic structure on  $\mathbb{O}_x^G$ , and consequently they all have the same dimension  $\frac{1}{2} \dim \mathbb{O}_x^G$  (see [Vog91, Corollary 5.20] for example). We also consider a complex semisimple orbit  $\mathbb{O}_a^K := \text{Ad}(K)a \subset \mathfrak{s}$ , which is closed. Note that  $a$  and  $h$  generate the same  $G$ -orbit,  $\mathbb{O}_a^G = \text{Ad}(G)a = \mathbb{O}_h^G$ .

Let us consider  $\text{ad } h$ -eigenspace decomposition  $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$  as in Equation (2.2). We put

$$\mathfrak{p} = \bigoplus_{k \geq 0} \mathfrak{g}_k = \mathfrak{l} \oplus \mathfrak{u}, \quad \text{where } \mathfrak{l} = \mathfrak{g}_0, \mathfrak{u} = \bigoplus_{k > 0} \mathfrak{g}_k. \quad (4.1)$$

Then  $\mathfrak{p}$  is a  $\theta$ -stable parabolic subalgebra, and  $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{k}$  is a parabolic in  $\mathfrak{k}$ . We denote  $P$  and  $Q$  the parabolic subgroups of  $G$  and  $K$  respectively corresponding to  $\mathfrak{p}$  and  $\mathfrak{q}$ . We follow the notation in §3.

**Theorem 4.2.** *Assume that  $x \in \mathfrak{s}$  is an even nilpotent element, and let  $\{x, h, y\}$  be a normal  $\mathfrak{sl}_2$  triple. After conjugation by  $K$ , we can assume  $\{x, h, y\}$  is a KS triple. Put  $a = \sqrt{-1}(x - y) \in \mathfrak{s}_{\mathbb{R}}$ . Then the asymptotic cone of  $\mathbb{O}_a^K$  is equal to*

$$\mathfrak{C}(\mathbb{O}_a^K) = \overline{\mathbb{O}_x^G \cap \mathfrak{s}} = \bigcup_{i=0}^{\ell} \overline{\mathbb{O}_{x_i}^K}, \quad (4.2)$$

where  $\{x = x_0, x_1, \dots, x_{\ell}\}$  is a complete set of representatives of the  $K$ -orbits in  $\mathbb{O}_x^G \cap \mathfrak{s}$ , and  $\{\mathbb{O}_{x_i}^K \ (0 \leq i \leq \ell)\}$  are Richardson orbits for a symmetric pair  $G/K$ .

*Proof.* Since  $x$  is even nilpotent by assumption, the  $K$ -orbit  $\mathbb{O}_x^K$  is a Richardson orbit corresponding to the  $\theta$ -stable parabolic  $\mathfrak{p}$  in (4.1). See [Noë06] for details. For  $1 \leq i \leq \ell$ , because  $x_i$  is a  $G$ -translate of  $x$ , they are all even nilpotent. Thus the same reasoning can be applied to the orbits  $\mathbb{O}_{x_i}^K$  which tells us that they are all Richardson.

Now let us consider  $a = \sqrt{-1}(x - y)$ . Then we calculate

$$\exp(t \text{ad } h)a = \sqrt{-1}(e^{2t}x - e^{-2t}y) = \sqrt{-1}e^{2t}(x - e^{-4t}y).$$

Therefore we get in  $\mathbb{P}(\mathfrak{g} \oplus \mathbb{C})$ ,

$$[\exp(t \text{ad } h)a \oplus 1] = [(x - e^{-4t}y) \oplus (-\sqrt{-1}e^{-2t})] \rightarrow [x \oplus 0] \in \kappa(\mathbb{P}(\mathfrak{g})) \quad (t \rightarrow \infty).$$

This proves that  $x \in \mathfrak{C}(\mathbb{O}_a^K)$  and hence  $\overline{\mathbb{O}_x^G} \subset \mathfrak{C}(\mathbb{O}_a^K)$  because  $\mathfrak{C}(\mathbb{O}_a^K)$  is a  $K$ -invariant closed set. By the same reason, we get  $\overline{\mathbb{O}_{x_i}^K} \subset \mathfrak{C}(\mathbb{O}_{a_i}^K)$ , where  $a_i$  is defined similarly as  $a$  by using  $x_i$  instead of  $x$ .

The semisimple elements  $a_i$ 's are in fact all conjugate to  $a$  by the adjoint action of  $K$ . This follows from the fact that representatives of the little Weyl group (the Weyl group of the restricted root system) can be chosen from the elements in  $K$  ([Kna02, Corollary 6.55]).

Thus we have proved that the right hand side is contained in the asymptotic cone  $\mathfrak{C}(\mathbb{O}_a^K)$ .

On the other hand, from Theorem 2.3, we clearly have

$$\mathfrak{C}(\mathbb{O}_a^K) \subset \mathfrak{C}(\mathbb{O}_a^G) \cap \mathfrak{s} \subset \overline{\mathbb{O}_x^G} \cap \mathfrak{s}.$$

Thus we get

$$\overline{\mathbb{O}_x^G \cap \mathfrak{s}} \subset \mathfrak{C}(\mathbb{O}_a^K) \subset \overline{\mathbb{O}_x^G} \cap \mathfrak{s}.$$

Note that  $\overline{\mathbb{O}_x^G \cap \mathfrak{s}}$  is a union of all irreducible components of  $\overline{\mathbb{O}_x^G} \cap \mathfrak{s}$  of maximal dimension  $\frac{1}{2} \dim \mathbb{O}_x^G$  (cf. Remark 4.3(1) below). Since  $\mathfrak{C}(\mathbb{O}_a^K)$  is equi-dimensional, it must coincide with  $\overline{\mathbb{O}_x^G \cap \mathfrak{s}}$ .  $\square$

**Remark 4.3.** (1) The inclusion  $\overline{\mathbb{O}_x^G \cap \mathfrak{s}} \subset \overline{\mathbb{O}_x^G} \cap \mathfrak{s}$  might be strict. For example, consider a symmetric pair  $(G, K) = (\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$  which is associated to  $\mathrm{U}(n, n)$ . Take the nilpotent  $G$ -orbit  $\mathbb{O}^G$  of Jordan type  $[3 \cdot 1^{2n-3}]$ . Then  $\overline{\mathbb{O}_x^G \cap \mathfrak{s}}$  consists of the  $K$ -orbits whose signed Young diagrams are

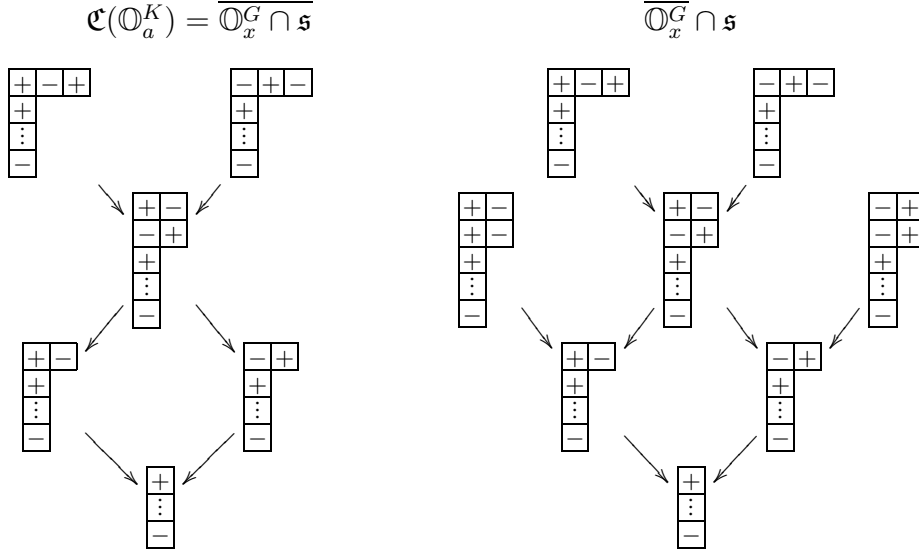
$$\begin{aligned} &[(+ - +) \cdot (+)^{n-2} \cdot (-)^{n-1}], \quad [(- + -) \cdot (+)^{n-1} \cdot (-)^{n-2}], \\ &[(+-) \cdot (-+) \cdot (+)^{n-2} \cdot (-)^{n-2}], \\ &[(+-) \cdot (+)^{n-1} \cdot (-)^{n-1}], \quad [(-+) \cdot (+)^{n-1} \cdot (-)^{n-1}], \\ &[(+)^n \cdot (-)^n], \end{aligned}$$

while the  $K$ -orbits  $[(+-)^2 \cdot (+)^{n-2} \cdot (-)^{n-2}]$  and  $[(-+)^2 \cdot (+)^{n-2} \cdot (-)^{n-2}]$  are not contained in the closure but contained in  $\overline{\mathbb{O}_x^G} \cap \mathfrak{s}$ . See the Hasse diagram of the closure relation below.

(2) The collection of  $\{\mathbb{O}_{x_i}^K \ (0 \leq i \leq \ell)\}$  is a set of Richardson orbits which are the moment map image of the conormal bundle of closed  $K$ -orbits in the fixed partial flag variety  $\mathfrak{B}_P$  through  $\theta$ -stable parabolics (not necessarily all of them). Let us denote a closed  $K$ -orbit in  $\mathfrak{B}_P$  by  $\mathcal{O}_i$  which corresponds to the Richardson orbit  $\mathbb{O}_{x_i}^K$ . If  $K_{x_i}$  is connected, the moment map  $\mu_i : T_{\mathcal{O}_i}^* \mathfrak{B}_P \rightarrow \overline{\mathbb{O}_{x_i}^K}$  is a resolution of the singularities (see Proposition 5.9 and §8.8 of [Jan04]).

Since  $a \in \mathfrak{s}_{\mathbb{R}}$  is a real hyperbolic element, it naturally defines a real parabolic subalgebra  $\mathfrak{p}_{\mathbb{R}}$ , which is the non-negative part of the  $\mathbb{Z}$ -grading similar to (4.1) with respect to  $\mathrm{ad} a$  instead of  $\mathrm{ad} h$ . Let us denote by  $P_{\mathbb{R}}$  the corresponding real parabolic subgroup of  $G_{\mathbb{R}}$ . A parabolically induced representation from a character  $\chi$  of  $P_{\mathbb{R}}$  is called the *degenerate principal series representation*, which is denoted by  $I_{P_{\mathbb{R}}}(\chi) = \mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} \chi$ .





**Corollary 4.4.** *We assume  $x \in \mathfrak{s}$  is even nilpotent and use the setting of Theorem 4.2. Let  $I_{P_{\mathbb{R}}}(\chi)$  be a degenerate principal series representation of  $G_{\mathbb{R}}$ , where  $P_{\mathbb{R}}$  is obtained from  $a \in \mathfrak{s}_{\mathbb{R}}$  as above. Then the associated variety of  $I_{P_{\mathbb{R}}}(\chi)$  is equal to the asymptotic cone  $\mathfrak{C}(\mathbb{O}_a^K)$  (see Equation (4.2)).*

*Proof.* It is known that the  $G$ -hull of the associated variety  $\mathcal{AV}(I_{P_{\mathbb{R}}}(\chi))$  is the closure of the Richardson  $G$ -orbit associated to  $P$ . Thus, by Theorem 4.2, we have  $\mathcal{AV}(I_{P_{\mathbb{R}}}(\chi)) \subset \mathfrak{C}(\mathbb{O}_a^K)$ . Note that the function ring  $\mathbb{C}[\mathcal{AV}(I_{P_{\mathbb{R}}}(\chi))]$  is asymptotically isomorphic to the space of  $K_{\mathbb{R}}$ -finite vectors in  $I_{P_{\mathbb{R}}}(\chi)$  as  $K_{\mathbb{R}}$ -modules. If  $\chi$  is trivial, we have

$$I_{P_{\mathbb{R}}}(\mathbf{1})|_{K_{\mathbb{R}}} \simeq \text{Ind}_{M_{\mathbb{R}}}^{K_{\mathbb{R}}} \mathbf{1} \simeq \mathbb{C}[\mathbb{O}_a^K], \quad M_{\mathbb{R}} = Z_{K_{\mathbb{R}}}(a).$$

Therefore, asymptotically  $\mathbb{C}[\mathcal{AV}(I_{P_{\mathbb{R}}}(\mathbf{1}))]$  and  $\mathbb{C}[\mathfrak{C}(\mathbb{O}_a^K)]$  are equal. So they must coincide with each other.  $\square$

**Remark 4.5.** The wave front set of  $I_{P_{\mathbb{R}}}(\chi)$  is known by the results in [BB99] (see also [Bar00]). Therefore, using Schmid-Vilonen's theorem [SV00], we basically know the associated variety of  $I_{P_{\mathbb{R}}}(\chi)$ . Here, in the corollary above, the emphasis is on the coincidence with the asymptotic cone.

The conclusion of Corollary 4.4 does not contain the even nilpotent element  $x$  explicitly. In fact, it is plausible to believe the conclusion is always true.

**Problem 4.6.** *Let  $a \in \mathfrak{s}_{\mathbb{R}}$  be a hyperbolic semisimple element and define the parabolic  $\mathfrak{p}_{\mathbb{R}}$  as above. Does the associated variety of the degenerate principal series  $I_{P_{\mathbb{R}}}(\chi)$  coincide with the asymptotic cone  $\mathfrak{C}(\mathbb{O}_a^K)$ ?*

Remark 4.7. (1) For a general  $a \in \mathfrak{s}_{\mathbb{R}}$ , it is no longer true that the asymptotic cone  $\mathfrak{C}(\mathbb{O}_a^K)$  is equal to the intersection of the closure of the Richardson orbit and  $\mathfrak{s}$ . For this, we refer to an example in [MT07, Example 3.8].

(2) There is a formula for the asymptotic  $K$ -support by T. Kobayashi, which is very close to the above problem. His formula ([Kob05, Theorem 6.4.3]) implies

$$\mathrm{AS}_K(I_{P_{\mathbb{R}}}(\chi)|_{K_{\mathbb{R}}}) = C^+ \cap \sqrt{-1} \mathrm{Ad}^*(K_{\mathbb{R}})(\mathfrak{m}_{\mathbb{R}})^{\perp},$$

where  $C^+$  denotes the closed Weyl chamber inside  $\sqrt{-1} \mathfrak{t}_{\mathbb{R}}^*$ . However, up to now, we do not know the exact relation of the above formula to our problem.

**Corollary 4.8.** *Suppose that  $x \in \mathfrak{s}$  is even nilpotent which satisfies*

- (1) *the fixed point subgroup  $K_x$  is connected,*
- (2)  *$\overline{\mathbb{O}}_x^K$  is normal,*
- (3)  *$\mathrm{codim} \partial \mathbb{O}_x^K \geq 2$ , where  $\partial \mathbb{O}_x^K = \overline{\mathbb{O}}_x^K \setminus \mathbb{O}_x^K$  is the boundary of  $\mathbb{O}_x^K$ .*

*Then the intersection  $\mathbb{O}_x^K \cap \mathfrak{s} = \mathbb{O}_x^K$  consists of a single  $K$ -orbit. If we take a KS triple  $\{x, h, y\}$  as above, the asymptotic cone of the semisimple orbit  $\mathbb{O}_a^K$  ( $a = \sqrt{-1}(x - y)$ ) is given by  $\mathfrak{C}(\mathbb{O}_a^K) = \overline{\mathbb{O}}_x^K$ . In this case, we have isomorphisms of algebra*

$$\mathbb{C}[T_{\mathcal{O}}^* \mathfrak{B}_P] \simeq \mathbb{C}[\mathbb{O}_x^K] \simeq \mathbb{C}[\overline{\mathbb{O}}_x^K],$$

*and, as  $K$ -modules, they are isomorphic to  $\mathbb{C}[\mathbb{O}_a^K]$ .*

*Proof.* We use the following lemma. Let us recall the notation  $m_{\tau}(\mathfrak{X})$  for the multiplicity defined in (2.1).

**Lemma 4.9.** *The following inequality holds.*

$$m_{\tau}(\mathbb{O}_a^K) \geq m_{\tau}(\mathfrak{C}(\mathbb{O}_a^K)) \geq m_{\tau}(\overline{\mathbb{O}}_{x_i}^K) \quad (\tau \in \mathrm{Irr}(K)).$$

*Proof.* Let us denote the annihilator ideal of  $\mathbb{O}_a^K$  by  $I = \mathbb{I}(\mathbb{O}_a^K) \subset \mathbb{C}[\mathfrak{s}]$ . Then we have  $\mathbb{C}[\mathbb{O}_a^K] \simeq \mathbb{C}[\mathfrak{s}]/\mathrm{gr} I$  as  $K$ -modules. Moreover, there is a surjective algebra morphism  $\mathbb{C}[\mathfrak{s}]/\mathrm{gr} I \twoheadrightarrow \mathbb{C}[\mathfrak{s}]/\sqrt{\mathrm{gr} I} = \mathbb{C}[\mathfrak{C}(\mathbb{O}_a^K)]$ . Since this morphism is  $K$ -equivariant, we have the following inequality

$$m_{\tau}(\mathbb{O}_a^K) \geq m_{\tau}(\mathfrak{C}(\mathbb{O}_a^K)) \quad (\tau \in \mathrm{Irr}(K)).$$

Since  $\overline{\mathbb{O}}_{x_i}^K$  in Theorem 4.2 is an irreducible component of  $\mathfrak{C}(\mathbb{O}_a^K)$ , we also have an inequality  $m_{\tau}(\mathfrak{C}(\mathbb{O}_a^K)) \geq m_{\tau}(\overline{\mathbb{O}}_{x_i}^K)$ .  $\square$

Let us return to the proof of the corollary.

By Theorem 4.2, we know  $\mathfrak{C}(\mathbb{O}_a^K)$  is the union of  $\overline{\mathbb{O}}_{x_i}^K$ 's. By Corollary 4.4,  $\mathfrak{C}(\mathbb{O}_a^K)$  is an associated variety of a degenerate principal series  $I_{P_{\mathbb{R}}}(\chi)$ . For a generic parameter  $\chi$ , the degenerate principal series representation is irreducible. So by Vogan's theorem ([Vog91, Theorem 4.6]), if there are more than two irreducible components of the associated variety, they must have a codimension one orbit in its boundary. But by the assumption, there is no such orbit, hence it must be irreducible.

The normality and the codimension-two condition imply the isomorphism  $\mathbb{C}[\overline{\mathbb{O}_x}] \xrightarrow{\sim} \mathbb{C}[\mathbb{O}_x]$ . Since  $K_x$  is connected the moment map  $\mu : T_{\mathcal{O}}^* \mathfrak{B}_P \rightarrow \overline{\mathbb{O}_x^K}$  is a resolution. By [Jan04, Proposition 8.9], we get  $\mathbb{C}[T_{\mathcal{O}}^* \mathfrak{B}_P] \simeq \mathbb{C}[\mathbb{O}_x^K]$ .  $\square$

## 5. EXAMPLE: SIEGEL PARABOLICS

Let  $G_{\mathbb{R}} = \mathrm{U}(n, n)$  and  $K_{\mathbb{R}} = \mathrm{U}(n) \times \mathrm{U}(n)$  a maximal compact subgroup. Then  $G = \mathrm{GL}_{2n}(\mathbb{C})$  is the complexification of  $G_{\mathbb{R}}$  and  $K = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  is block diagonally embedded into  $G$ .  $(G, K)$  is a symmetric pair. The Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$  is given as follows.

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A, D \in \mathrm{M}_n(\mathbb{C}) \right\}, \quad \mathfrak{s} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in \mathrm{M}_n(\mathbb{C}) \right\}$$

Let us consider a nilpotent element

$$x = \begin{pmatrix} 0 & 1_n \\ 0 & 0 \end{pmatrix} \in \mathfrak{s}.$$

If we put  $y = {}^t x$  and  $h = [x, y]$ , then  $\{x, h, y\}$  constitute a KS triple. Note that, in this case, the complex conjugation  $\sigma$  with respect to the real form  $\mathfrak{g}_{\mathbb{R}}$  is given by

$$\sigma(X) = -I_{n,n} {}^t \overline{X} I_{n,n} \quad (X \in \mathfrak{g}), \quad I_{n,n} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}.$$

We can check  $\sigma(x) = {}^t x = y$  directly.

The nilpotent element  $x$  generates a nilpotent  $G$ -orbit  $\mathbb{O}_x^G$  which has Jordan type  $[2^n]$ . Consequently  $x$  is even nilpotent. There are  $(n+1)$  nilpotent  $K$ -orbits in  $\mathbb{O}_x^G \cap \mathfrak{s}$ , which are  $\mathbb{O}_{p,q}^K = [(+-)^p(-+)^q]$  ( $p, q \geq 0, p+q = n$ ) in the notation of signed Young diagram (see [CM93], for example).

Put  $a = \sqrt{-1}(x - y) \in \mathfrak{s}_{\mathbb{R}}$ . Theorem 4.2 tells us that

$$\mathfrak{C}(\mathbb{O}_a^K) = \bigcup_{p+q=n} \overline{\mathbb{O}_{p,q}^K}.$$

Let us interpret the meaning of this identity in terms of the representation theory of  $G_{\mathbb{R}}$ .

First, let us see the function ring  $\mathbb{C}[\mathbb{O}_a^K]$ . Put  $M = Z_K(a)$ , the stabilizer of  $a$  in  $K$ . Then clearly  $M = \Delta \mathrm{GL}_n(\mathbb{C})$ , the diagonal embedding of  $\mathrm{GL}_n(\mathbb{C})$  into  $K = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ . Thus we have

$$\mathbb{C}[\mathbb{O}_a^K] = \mathbb{C}[K/M] = \mathbb{C}[K]^M \simeq \mathrm{Ind}_M^K \mathbf{1}_M, \quad (5.1)$$

where the last isomorphism is an isomorphism as  $K$ -modules, and  $\mathbf{1}_M$  denotes the trivial representation of  $M$ . Thus we have

$$\mathbb{C}[\mathbb{O}_a^K] \simeq \bigoplus_{\rho \in \mathrm{Irr}(\mathrm{GL}_n)} \rho \otimes \rho^* \quad (\text{as a } K \simeq \mathrm{GL}_n \times \mathrm{GL}_n\text{-module}), \quad (5.2)$$

which is a multiplicity free  $K$ -module. This is isomorphic to  $\mathbb{C}[\mathfrak{C}(\mathbb{O}_a^K)]$  as a  $K$ -module by [NOZ06, Theorem 3.1].

On the other hand, by explicit calculation using the technique in [Nis00] (also see [Nis04]), we have

$$\mathbb{C}[\overline{\mathbb{O}_{p,q}^K}] \simeq \bigoplus_{\alpha \in \mathcal{P}_p, \beta \in \mathcal{P}_q} \rho_{\alpha \odot \beta} \otimes \rho_{\alpha \odot \beta}^*.$$

However, we have the following

**Proposition 5.1.** *For any  $p, q \geq 0$  satisfying  $p + q = n$ , there are isomorphisms of  $K$ -modules*

$$\mathbb{C}[\mathbb{O}_{p,q}^K] \simeq \mathbb{C}[\mathfrak{C}(\mathbb{O}_a^K)] \simeq \mathbb{C}[\mathbb{O}_a^K],$$

where the first isomorphism is also a morphism of algebras induced by the open embedding  $\mathbb{O}_{p,q}^K \hookrightarrow \mathfrak{C}(\mathbb{O}_a^K)$ .

Let us denote  $M_{\mathbb{R}} = Z_{K_{\mathbb{R}}}(a) = \Delta U(n)$ , and  $L_{\mathbb{R}} = Z_{G_{\mathbb{R}}}(a) \simeq \mathrm{GL}_n(\mathbb{C})$ . The semisimple element  $a$  naturally defines a maximal parabolic subgroup  $P_{\mathbb{R}} = L_{\mathbb{R}} N_{\mathbb{R}}$ , where  $N_{\mathbb{R}}$  is a suitably chosen unipotent radical. Note that  $A_{\mathbb{R}} = \exp \mathbb{R}a$  is contained in the center of  $L_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{C})$  as the radial part of the complex torus. We consider a degenerate principal series representation induced from a one dimensional character of  $P_{\mathbb{R}}$  (unnormalized induction)

$$I(\nu) := \mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} (|\det|^{\nu+2n} \otimes \mathbf{1}_{N_{\mathbb{R}}}), \quad (\nu \in \mathbb{C}),$$

where  $\det$  is the determinant character of  $L_{\mathbb{R}} = \mathrm{GL}_n(\mathbb{C})$  and the induced character is trivial on  $N_{\mathbb{R}}$ . Then we have

$$I(\nu)|_{K_{\mathbb{R}}} \simeq \mathrm{Ind}_{M_{\mathbb{R}}}^{K_{\mathbb{R}}} \mathbf{1}_{M_{\mathbb{R}}} \simeq \bigoplus_{\rho \in \mathrm{Irr}(U(n))} \rho \otimes \rho^*.$$

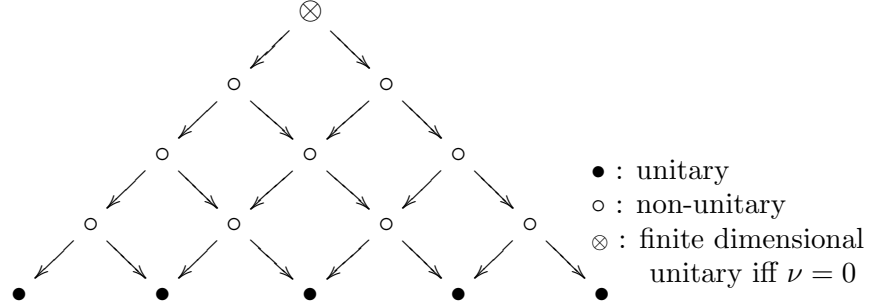
Comparing this with (5.2) and (5.1), we conclude that  $\mathbb{O}_a^K$  or  $\mathfrak{C}(\mathbb{O}_a^K)$  carries information of  $K$ -types of degenerate principal series  $I(\nu)$ .

**Theorem 5.2** (Sahi, Lee, Johnson, Wallach, ...). *Assume that  $\nu \geq 0$  is even. Then the degenerate principal series  $I(\nu)$  contains precisely  $(n+1)$  irreducible subrepresentations  $\pi_{p,q}(\nu)$  ( $p, q \geq 0$ ,  $p+q=n$ ), which are unitary. If  $\nu > 0$ , then these are only unitarizable irreducible constituents of  $I(\nu)$ .*

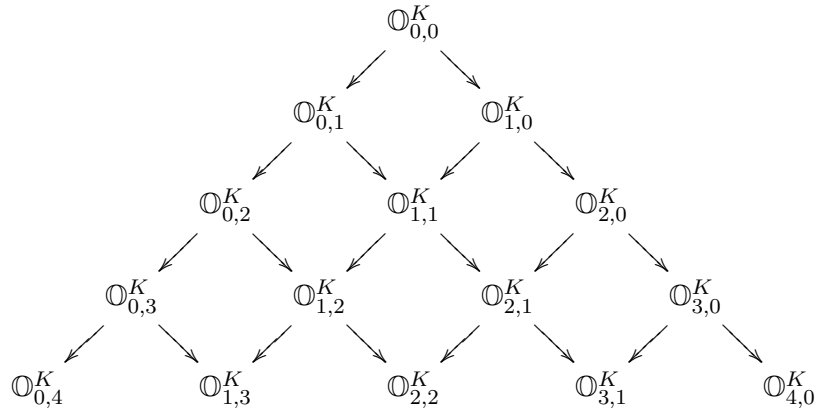
**Remark 5.3.**  $I(\nu)$  is reducible if and only if  $\nu$  is an even integer. If  $\nu \geq 0$  (and even), then the Hasse diagram of subquotients of  $I(\nu)$  is given below (see [Lee94, §§7&9] and also [Sah93]).

If  $\nu = 0$ , then  $I(\nu)$  contains the trivial representation. In general  $I(\nu)$  ( $\nu \geq 0$ ) contains a finite dimensional representation as a unique irreducible subrepresentation.

If  $\nu = -n$ , then  $I(-n)$  is a direct sum of  $(n+1)$  irreducible unitary representations  $\{\pi_{p,q}(-n) \mid p+q=n\}$ , which are derived functor modules  $A_{\mathfrak{p}_{p,q}}$  (see [MT07]). The representations  $\pi_{p,q}(\nu)$  ( $p+q=n$ ) are translation (or coherent continuation) of these derived functor modules.



$n = 4$  : Hasse diagram of submodules of  $I(\nu)$  ( $\nu \in 2\mathbb{Z}_{\geq 0}$ )



Hasse diagram of associated varieties

**Corollary 5.4.** *The associated variety of  $I(\nu)$  is equal to  $\mathfrak{C}(\mathbb{O}_a^K) = \bigcup_{p+q=n} \overline{\mathbb{O}_{p,q}^K}$ . The associated cycle of the largest constituents  $\pi_{p,q}(\nu)$  ( $p+q = n$ ) is given by  $\mathcal{AC} \pi_{p,q}(\nu) = [\overline{\mathbb{O}_{p,q}^K}]$  with multiplicity one.*

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